数理解析研究所講究録
240巻 1975年 31-47

Tricanonical map of a certain class of surfaces

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INTRODUCTION. Let S be an projective algebraic surface defined over the complex number field C. We let K_S denote the canonical bundle of S, and mK_S its m-th tensor power. Consider the rational map Φ_{mK_S} associated with the complete linear system $|mK_S|$ (pluricanonical map). S is called of general type if $\Phi_{mK_S}(S)$ is a surface for $m \gg 0$. Putting $R = \sum_{m=0}^{\infty} H^0(S, \mathcal{O}(mK_S))$, the projective scheme $X = \mathrm{Proj}\,(R)$ is called an (abstract) canonical model of S. It is known that the natural rational map

$$X \to \Phi_{mK_S}(S) \simeq \mathrm{Proj}(\mathcal{S} H^0(S, mK_S))$$

is an isomorphism for $m \gg 0$, and that the rational map $S \to X$ is a birational morphism.

In this paper, we are concerned with the algebraic surface S whoes numerical characters are: $K_S^2 = 1$, $p_g = 0$, where p_g is the geometric genus. We shall prove the following

MAIN THEOREM. Φ_{3K_S} is birational.

Let S be a minimal surface of general type.

By this theorem Bombieri's result about the birationality of pluricanonical maps [3] is sharpened as follows:

Φ_{3K_S} is birational except in the following cases:

a) $K_S^2=1$, $p_g=2$, where $\Phi_{3K}(S)$ is rational;

b) $K_S^2=2$, $p_g=3$, where $\Phi_{3K}(S) = P^2$;

c) $K_S^2=2$, $p_g=0$. (It is expected that the case c) does not occur.)

1. Generalities. In this section we review the well-known results that are used in our proof.

THEOREM A (algebraic index theorem). Let S be an algebraic surface. The intersection numbers for pairs of divisors define a quadratic form Q on the numerical divisor group $\mathrm{Num}(S)$. Q is non-degenerate and has one and only one positive eigenvalue.

THEOREM B. Let S be a minimal surface of general type. Then for any irreducible curve C on S we have

$$K_S C \geq 0.$$

Moreover, the curves C satisfying $K_S C=0$ form a finite set and are numerically independent of each other.

Let D be an effective divisor on S. We say that D is numerially connected (or 1-connected) if for any non-trivial decomposition $D=D_1+D_2$, $D_i>0$, we have $D_1 D_2>0$.

THEOREM C (Ramanujam). If an effective divisor D is 1-connected, then $\dim H^0(D,\mathcal{O}_D)=1$.

Let S be a minimal surface of general type and X a canonical model of S. The natural map $\varpi_0: S \to X$ is a minimal resolution of singularities of X. X is a normal surface with a finite number of rational double points. Let $\mathfrak{m}$ be the maximal ideal of a rational double point P on X. $\varpi_0^* \mathfrak{m}$ is an invertible sheaf that defines a divisor Z. Z is called a fundamental cycle. Z is a sum of irreducible curves C_i such that $C_i K_S=0$. Conversely such curves are contained in some fundamental cycles.

PROPOSITION 1 (Artin [1] [2]).

(i) An effective divisor Z on S is a fundamental cycle if and only if Z is a maximal cycle with

$K_S Z=0, \quad Z^2=0.$

(ii) We have $K_S = \varpi_0^* K_X$, where K_X is a line bundle on X.

(iii) For two line bundles δ_1 and δ_2 on a fundamental cycle, δ_1 and δ_2 are isomorphic to each other if $\deg \delta_1 = \deg \delta_2$.

We shall denote the numerical equivalence by the symbol $\sim$. Thus $D \sim D'$ means that D is numerically equivalent to D'.

Let S be a minimal surface of general type, Z a fundamental cycle, $\varpi : \tilde{S} \to S$ a blowing up, and E the exceptional curve on $\tilde{S}$.

LEMMA 1 (Bombieri's connectedness theorem).

(i) If D is effective and $D \sim mK_S$ $(m \geq 1)$, then D is 1-connected.

(ii) If D is effective and $D \sim mK_S$ $(m \geq 2)$, then for any decomposition $D=D_1+D_2$, $D_i > 0$, $D_i K_S > 0$, we have $D_1 D_2 \geq 3$, except if $K_S^2=1$ or 2, and D_1 or $D_2 \sim K_S$.

(iii) If $D \sim mK_S - Z$ $(m \geq 1)$, then D is 1-connected.

(iv) If $D \sim m\varpi^* K_S - 2E$ $(m \geq 1)$, then D is 1-connected except if $K_S^2 =1$, $m=2$, $D=D_1+D_2$, $D_1 \sim D_2 \sim \varpi^* K_S - E$.

PROOF. For the convenience of the reader, we shall give a proof following Bombieri [3]. We discuss in the rational numerical group $\mathrm{Num}_Q(S) = \mathrm{Num}(S) \otimes Q$. We let $t=K_S^2$.

(i) Let $D=D_1+D_2$, $D_i > 0$ be a decomposition of D. We have $0 \leq r=D_1 K_S \leq mt$. Hence

$$D_1 = \frac{r}{t} K_S + \xi, \qquad \xi \cdot K_S = 0,$$

$$D_2 = \frac{1}{t}(mt-r) K_S - \xi,$$

$$D_1 D_2 = \frac{1}{t} r(mt-r) - \xi^2.$$

From Theorem A we infer $\xi^2 < 0$ except if $\xi \sim 0$. $D_1 D_2 \leq 0$ implies that $r=0$ or $=mt$, and $\xi \sim 0$. Therefore we have D_1 or $D_2 \sim 0$, i.e.,

D_1 or $D_2 = 0$.

(ii) In this case $1\leqq r\leqq mt-1$. Suppose $2\leqq r\leqq mt-2$. Then

$$D_1D_2=\frac{1}{t}\cdot r(mt-r)-\xi^2\geqq\frac{1}{t}\cdot 2(mt-2) = 2m-\frac{2}{t}.$$

Since $mt\geqq 4$, $2m-\frac{2}{t}\geqq 3$. Next consider the case $r=D_1K_S=1$. Then $D_1{}^2=\frac{1}{t}+\xi^2=$ odd. Hence, unless $t=1$ and $\xi\sim 0$, $D_1{}^2\leqq -1$.

(iii) Let $D=D_1+D_2$, $D_i>0$. If $D_iK_S>0$ and if $D_i\not\sim K$, we get $D_1(D_2+Z)\geqq 3$, $D_2(D_1+Z)\geqq 3$. Summing up these, we have $2D_1D_2+DZ\geqq 6$. On the other hand, $DZ=-Z^2=2$. Hence $D_1D_2\geqq 2$. If $D_iK_S>0$ and if $D_1\sim K_S$, we have $D_1D_2=K_S((m-1)K_S-Z)=(m-1)\geqq 1$. If $D_iK_S=0$, we have $D_1{}^2\leqq -2$, $D_2\sim mK_S+\xi$, $\xi^2\leqq 0$. Hence $2D_1D_2=(mK_S-Z)^2-D_1{}^2-D_2{}^2\geqq -\xi^2$. $D_1D_2=0$ implies that $\xi\sim 0$ and that $D_2\sim mK_S$, $D_1\sim -Z$, a contradiction.

(iv) Let $D=D_1+D_2$, $D_i>0$, and $v=D_1E$. Then we have

$$D_1\sim\varpi^*D_1'-vE,\quad D_1'\sim\frac{r}{t}K_S+\xi.$$

$$D_2\sim\varpi^*D_2'-(2-v),\quad D_2'\quad (m-\frac{r}{t})K_S-\xi.$$

Note that D_i' is an effective divisor on S. So $D_1'D_2'\geqq 1$, and

$$D_1D_2\geqq 1-v(v-2)E^2 = 1+v(v-2).$$

Hence $D_1D_2>0$ unless $v=1$. Suppose $D_i'K_S>0$ and $v=1$. In this case $D_1'D_2'\geqq 3$ unless D_1' or $D_2'\sim K_S$. Hence $D_1D_2>1$. Suppose $D_1'\sim K_S$. Then $D_1\sim\varpi^*K_S-E$, $D_2\sim\varpi^*(m-1)K_S-E$, $D_1D_2=(m-1)t-1$. Thus $D_1D_2=0$ if and only if $m=2$, $t=K_S{}^2=1$, $D_i\sim\varpi^*K_S-E$. Finally suppose that $D_1'K_S=0$. Then $D_1'^2\leqq -2$, and so $D_1'D_2'=-D_1'^2\geqq 2$. Hence, $D_1D_2=D_1'D_2'+v(v-2)\geqq D_1'D_2'-1\geqq 1$. Q.E.D.

THEOREM D. Let S be as in Lemma 1 and let $\mathcal{L}$ an invertible sheaf such that $\mathcal{L}^n$ is spanned by its global sections and has three algebraically independent sections for $n\gg 0$. Then we have

$$H^1(S, \mathcal{L}^{-1}) = 0.$$

For the proof, see Mumford [ ].

COROLLARY. If a divisor $M \sim mK_S$ $(m \geq 2)$, then $H^1(S,M)=0$.

PROOF. For $n \gg 0$, consider the exact sequence

$$0 \to H^0(\tilde{S}, n\varpi^*M - E) \to H^0(\tilde{S}, n\varpi^*M) \to H^0(E, n\varpi^*M) \to H^1(\tilde{S}, n\varpi^*M - E).$$

By the Serre duality theorem we have

$$\dim H^1(\tilde{S}, n\varpi^*M - E) = \dim H^1(\tilde{S}, 2E - (n-1)\varpi^*M).$$

Since $D \sim (n-1)\varpi^*M - 2E$ is 1-connected by Lemma 1, $H^1(\tilde{S}, 2E - (n-1)\varpi^*M))$ $=0$. Thus $H^1(\tilde{S}, n\varpi^*M - E) = 0$. Hence $|nM|$ has no base point ($n \gg 0$). Now apply the theorem. Q.E.D.

2. Numerical Godeaux surfaces. We call S a numerical Godeaux surface if it is a minimal surface of general type with numerical characters $K_S^2=1$, $p_g=0$, where p_g denotes the geometric genus $\dim H^2(S,\mathcal{O}_S)$. The following theorem is classical (see [7]).

THEOREM 1. For a surface of general type S, we have $p_g \geq q = \dim H^1(S,\mathcal{O}_S)$.

In what follows, we denote by S a numerical Godeaux surface. As a corllary to Theorem C and Theorem 1, we obtain the following

LEMMA 2. If an effective divisor D is 1-connected, then $H^1(S,\mathcal{O}_S(-D))=0$.

LEMMA 3. If $D \sim K_S$, we have $\dim H^0(S,\mathcal{O}_S(D)) \leqq 1$.

PROOF. By the Riemann-Roch theorem and the corollary to Theorem D, we have $\dim H^0(S,\mathcal{O}_S(2D))=2$. Suppose that $\dim H^0(S,\mathcal{O}_S(D)) \geqq 2$. Then we have $\dim H^0(S,\mathcal{O}_S(2D)) \geqq 3$, a contradiction. Q.E.D.

LEMMA 4. If an effective divisor $D \sim K$, we have $H^1(S,\mathcal{O}_S(D)) =0$.

PROOF. We may assume that D is not linearly equivalent to K_S. Since $\dim H^0(S,\mathcal{O}_S(D)) \leqq 1$ by Lemma 3 and since $\chi(S,\mathcal{O}_S(D))=\chi(S,\mathcal{O}_S(K_S))=1$, we have

$$\begin{aligned}\dim H^1(S,\mathcal{O}_S(D)) &= -\chi(S,\mathcal{O}_S(D))+\dim H^0(S,\mathcal{O}_S(D))+\dim H^2(S,\mathcal{O}_S(D)) \\ &\leqq \dim H^2(S,\mathcal{O}_S(D))=0.\end{aligned}$$

Q.E.D.

Remark. By the vanishing of q, we know that the linear equivalence coincides with the algebraic equivalence. Hence if $\dim H^0(S,\mathcal{O}_S(D))=1$ $(D>0)$, then there exists no effective divisor algebraically equivalent to D except D itself. Lemma 4 implies that for any non-zero $\tau \in H^2(S,Z)_{tor}$, there is one and only one effective divisor D' which is algebraically equivalent to $K_S+\tau$.

LEMMA 5. Let D be an effective divisor and assume that $\dim|D| \geqq 1$. Then $DK_S \geqq 2$.

PROOF. By Theorem B, we may assume that $|D|$ is fixed part free. Hence we have $D^2 \geqq 0$. This implies that $DK_S \geqq 2$ or $D \sim K_S$. But the latter case is impossible (see the Remark above).

Q.E.D.

We let M denote the generic member of the moving part of the dicanonical system $|2K_S|$, F the fixed part of $|2K_S|$. Thus $|2K_S| = |M|+F$. From the Riemann-Roch theorem we infer that $|M|$ is composed of a pencil over a projective line P^1.

LEMMA 6. If M is generically chosen, M is reduced and irreducible. Morever, M and F satisfy one of the following numerical conditions:

a) $F=0$;

b) $FK_S=0$, $F^2=-2$, $M^2=2$, $MF=2$;

c) $FK_S=0$, $F^2=-4$, $M^2=0$, $MF=4$.

PROOF. First we note that $MK_S \geqq 2$ in virtue of Lemma 5; in other words, $FK_S=0$. Suppose generic M admits a non-trivial decomposition $M=M_1+M_2$, $M_i > 0$. Since the M_i can <u>move</u>, we have $M_iK_S \geqq 2$, which is a contradiction. From $FK_S=0$ follows $F^2 \leqq 0$ unless $F=0$. On the other hand,

$$F^2=-FM=M^2-2MK_S \geqq -2MK_S=-4. \qquad \text{Q.E.D.}$$

Rmark. If D is a divisor numerically equivalent to $2K_S$, then $\dim |D|=1$. The similar argument in Lemma 6 is valid for $|D|$ in place of $|2K_S|$. Thus the fixed part D_o of D satisfies $D_oK_S=0$, and the generic member of the moving part is an irreducible curve.

From the lemma above we obtain the following

COROLLARY. Let X denote the canonical model of S. $|2K_X|$ has no fixed part and its generic member is irreducible.

Let $\hat{M}$ be a generic member of $|2K_X|$, and consider the natural exact sequence

$$0 \to \mathcal{O}_X(-2K_X) \to \mathcal{O}_X \to \mathcal{O}_{\hat{M}} \to 0.$$

Since X has at most a finite number of rational singularities and $\mathcal{O}_X(-2K_X)$ is an invertible sheaf, we have canonical isomrphisms

$$H^i(X, \mathcal{O}_X) \xrightarrow{\approx} H^i(S, \mathcal{O}_S),$$
$$H^i(X, \mathcal{O}_X(-2K_X)) \xrightarrow{\approx} H^i(S, \mathcal{O}_S(-2K_S)).$$

Hence $\hat{M}$ is an irreducible curve of virtual genus $\pi(\hat{M})=4$.

COROLLARY. F is a disjoint union of fundamental cycles.

PROOF. Let Z denote fundamental cycle such that $Z \cap F \neq \phi$. Since $2K_S$ is trivial on Z, we have $\operatorname{supp} F \supset Z$. If $F^2=-2$, then F is a fundamental cycle (see Proposition 1). Assume that $F^2=-4$. It is sufficient to prove that F is not connected in this case. Since $M^2=0$, M is base point free. If M is generic, M is a non-singular curve of genus 2. If F is connected, $\hat{M} = \varpi_0(M)$ has a 4-ple point. Hence $\pi(\hat{M})$ 2+3=5. This contradicts the above corollary. Q.E.D.

Now we proceed to the study of the tricanonical system $|3K_S|$.

LEMMA 7 (Bombieri). If $CK_S=0$, then C is not contained in the fixed part of $|3K_S|$.

PROOF. Note that

$$\dim H^0(S,\mathcal{O}_S(2K_S-Z))\geq \dim H^0(S,\mathcal{O}_S(2K_S))-\dim H^0(Z,\mathcal{O}_Z)=1,$$

and a fortiori $|2K_S-Z|$ contains a effective divisor D. D is numerically connected (see Lemma 1). So $\dim H^1(S,\mathcal{O}_S(Z-2K))=0$. If C is contained in the fixed part G of $3K_S$, G must also contain the fundamental cycle Z to which C belongs. Hence we have the canonical isomorphism

$$H^0(S,\mathcal{O}_S(3K_S-Z))\xrightarrow{\approx} H^0(S,\mathcal{O}_S(3K_S)).$$

This implies that

$$\dim H^1(S,\mathcal{O}_S(3K_S-Z))=\dim H^1(S,\mathcal{O}_S(Z-2K_S))\neq 0.$$

This is absurd. Q.E.D.

LEMMA 8. $|3K_S|$ is not composed of a pencil.

PROOF. Suppose the contrary. $\Phi_{3K_S}(S)$ is a space curve of $\deg\geq 3$. Let $\hat{S}\xrightarrow{\varpi}S$ be the resolution of the base points of $3K_S$. The moving part of $|3\varpi^*K_S|$ is generically a union of at least 3 components. Hence $|3K_S|$ contains at least 3 irreducible components each of which can <u>move</u>. Therefore $|3K_S|$ admits a decomposition $D_1+D_2+D_3$ such that $D_iK_S\geq 2$. This contradicts the equality $3K_S{}^2=3$. Q.E.D.

PROPOSITION 2. $|3K_S|$ has no fixed part.

PROOF. Suppose the fixed part $G>0$. From Lemma 7 we infer that $GK_S>0$. $(3K_S-G)K_S\geq 2$, since otherwise $\dim|3K_S-Z|<1$. Note that $3K-G\not\sim K_S$. This leads to the inequality $(3K_S-G)G\geq 3$. Thus

$$(3K_S-G)^2=(3K_S-G)3K_S-(3K_S-G)G\leqq 3.$$

This implies the ablurd conclusion that Φ_{3K_S} is a birational map of S onto a quadratic or a cubic hypersurface P^3.

Q.E.D.

PROPOSITION 3. Let M denote the moving part of $|2K_S|$. If M is generic, M contains no base points of $|3K_S|$.

PROOF. First we consider the case where $F^2=0$, -2 and M is non-singular. In view of the exact sequence

$$0 \to H^0(S, \mathcal{O}_S(3K_S)) \to H^0(M, \mathcal{O}_M(3K_S|_M)) \quad 0,$$

we have only to prove that $3K_S|_M = K_M + F|_M$ is free from base points. This is, however, a classical fact. Next let us consider the case $F^2=-4$. In this case M is base point free, so generic M does not contain a base point. Finally we consider the case $F=0$ and M has a double point P which is a unique base point of $|M|$. Let $\varpi: \tilde{S} \to S$ denote the quadric transformation at P and E the associated exceptional curve on $\tilde{S}$. $\tilde{M}=\varpi^* M-2E$ is a non-singular curve of genus 3. The sequence of sheaves

$$0 \to \mathcal{O}_{\tilde{S}}(\varpi^* K_S-2E) \to \mathcal{O}_{\tilde{S}}(3\varpi^* K_S) \to \mathcal{O}_{\tilde{M}}(3\varpi^* K_S|_{\tilde{M}}) \to 0.$$

is exact and we obtain an injection $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3\varpi^* K_S)) \hookrightarrow H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(3\varpi^* K_S))$. Since $\dim H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(3\varpi^* K_S)) = \dim H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3\varpi^* K_S))=4$, this is an isomorphism. On the other hand, $|K_{\tilde{M}}+E|$ is base point free. Hence $|3\varpi^* K_S|$ is base point free on $\tilde{M}$. This implies that $|3K_S|$ has no base point on M.

Q.E.D.

COROLLARY. Let $\hat{M}$ be the generic member of $|2K_X|$. Then $|3K_X|$ has no base point on $\hat{M}$ and $\Phi_{3K}|_{\hat{M}}: \hat{M} \to \Phi_{3K}(\hat{M}) \subset P^3$ is a holomorphic mapping.

Remark. $\Phi_{3K_S}(M)$ is not a plane curve, so deg $\Phi_{3K_S}(M)=3$ or 6.

If deg $\Phi_{3K}(M)=3$, $\Phi_{3K}(M)\cong P^1$ and $\Phi_{3K}|_M$ is a double covering. If deg $\Phi_{3K}(M)=6$, $\Phi_{3K}|_M$ is a birational morphism.

PROPOSITION 4. If deg $\Phi_{3K}(M)=6$, $\Phi_{3K}(M)$ is a complete intersection of type (2,3). Moreover $\Phi_{3K}|_{\hat M}$ is an isomorphism.

PROOF. Consider the exact sequences

$$0 \to \mathcal{O}_X(4K_X) \to \mathcal{O}_X(6K_X) \to \mathcal{O}_{\hat M}(6K_X) \to 0,$$
$$0 \to \mathcal{O}_X(7K_X) \to \mathcal{O}_X(9K_X) \to \mathcal{O}_{\hat M}(9K_X) \to 0.$$

By the Riemann-Roch theorem, dim $H^0(\hat M, \mathcal{O}_{\hat M}(6K_X))=9$. On the other hand, dim $S^2H^0(\hat M, \mathcal{O}_{\hat M}(3K_X))=10$. Hence there exists a quadric Q which contain $\Phi_{3K_X}(\hat M)$. Such quadric is unique. In fact, if two quadrics contain a curve C, deg $C \leqq 4$. Next note that dim $H^0(\hat M, \mathcal{O}_{\hat M}(9K_X))=15$. Since there are only four <u>independent</u> cubic surfaces which contain Q, there exists a cubic surface R which contains $\Phi_{3K_X}(\hat M)$ and does not contain Q. Thus, since deg $R\cdot Q$ $=$deg $\Phi_{3K_X}(\hat M)=6$, we have $\Phi_{3K_X}(\hat M)=R\cdot Q$. We have

$$\chi(\Phi_{3K}(\hat M), \mathcal{O})= -3.$$

Accordingly $\pi(\Phi_{3K}(\hat M))=\pi(\hat M)=4$. Since $\Phi_{3K}|_{\hat M}$ is a birational morphism, this means $\Phi_{3K}|_M$ is an isomorphism. Q.E.D.

We shall end this section by the following

LEMMA 9. Let $\varpi:\tilde S\to S$ be a resolution of the base points of $|3K_S|$ and let $\tilde\Phi$ denote the associated holomorphic mapping. If $\tilde\Phi$ maps an irreducible curve C onto a point, then C is an exceptional curve on $\tilde S$ or an irreducible component of fundamental cycles.

PROOF. Let $\tilde C$ be an irreducible curve on $\tilde S$ such that $\hat C=\varpi_0\circ\varpi(\tilde C)$ is an irreducible curve with $\hat C K_X>0$. Suppose that $\tilde\Phi(\tilde C)$ is a point. This is equivalent to the equality

$$\dim H^0(S, \mathcal{O}_S(3K_S - \varpi(\widetilde{C}))=3.$$

From Lemma 5, we infer $\varpi(\widetilde{C})K_S=1$. The generic member D of $|3K_S-\varpi(\widetilde{C})|$ is an effective divisor with $DK_S=2$, $D^2 \leqq 2$. Note that D is not composed of a pencil. Let $|D'|$ be the moving part of $|D|$. We have $D'K_S=2$ and $D'^2=2$. Thus $\Phi_{D'}: S \to P^2$ is a double covering, and D' is a non-singular hyperelliptic curve of genus 3. Therefore $K_S|_{D'} \cong D'|_{D'}$. Consider the exact sequence:

$$0 \to H^0(S, \mathcal{O}_S(-K_S)) \to H^0(S, \mathcal{O}_S(D'-K_S)) \to H^0(D', \mathcal{O}_{D'}(D'-K_S)) \to 0.$$

Since $\dim H^0(D', \mathcal{O}_{D'}(D'-K_S))=1$, there exists an effective curve $D'' \in |D'-K|$. D" satisfies $D''^2=-1$, $D''K_S=1$. Assume that D" is 1-connected. Then $H^1(S, \mathcal{O}_S(-D''))=0$, and $H^1(S, \mathcal{O}_S(K_S+D''))$ $=H^1(S, \mathcal{O}_S(D'))=0$. This leads to the equality $\dim H^0(S, \mathcal{O}_S(D'))$ $=1$, a contradiction. Next suppose that D" is not 1-connected. D" admits a decomposition $D''=D_1''+D_2''$ with $D_1''K_S=1$, $D_2''K_S=0$, $D_2''^2 \leqq -2$, $D_1''D_2''=0$. Then we infer that $D_1'' \sim K_S$. $D'-K_S-D_1''$ is effective. Hence we have

$$\dim H^0(S, \mathcal{O}_S(D')) = \dim H^0(S, \mathcal{O}_S(K_S+D_1''))=2.$$

This is a contrandiction. Q.E.D.

3. Proof of the Main Theorem. We let $\Phi = \Phi_{3K_S}: S \to P^3$ and $\hat{\Phi} = \Phi_{3K_X}: X \to P^3$. Thus $\Phi = \hat{\Phi} \circ \varpi_0$. Let m and d denote the mapping degree of Φ and the degree of the hypersurface $\Phi(S) \subset P^3$, respectively. In order to prove our main theorem it suffices to deny each of the following possibilities:

(a) $d=2$;

(b) $d=3$ and $m=2$;

(c) $d=3$ and $m=3$;

(d) $d=4$ and $m=2$.

We have proved that $\hat{M}$ is an irreducible curve of virtual genus 4 for a generic member $\hat{M} \in |2K_X|$. First we prove the following

LEMMA 10. If $\hat{\Phi}(\hat{M})$ is a projective line of degree 3 embeded in P^3, then $m \geq 4$.

PROOF. Note that, if $\hat{\Phi}(M)$ is of degree 3, m is even. Suppose $m=2$. Let $\tilde{X} \xrightarrow{\varpi} X$ be a resolution of the base points of $|3K_X|$ and $\tilde{\Phi}: X \to P^3$ the associated holomorphic mapping. From Lemma 9 we infer that $\tilde{\Phi}^*(\hat{\Phi}(\hat{M})) = \varpi^*(\hat{M}) + D$ where D is a divisor whose support lies on the exceptional curves of X. Hence $\tilde{\Phi}^*(\hat{\Phi}(\hat{M}))|\varpi^*(\hat{M}) \cong \varpi^*(\hat{M})|\varpi^*(\hat{M})$.

But this is impossible, because

$$1 = \dim H^0(\hat{M}, \mathcal{O}(\hat{M})) \geq \dim H^1(\varpi^*(\hat{M}), \mathcal{O}_{\varpi^*(\hat{M})}(\varpi^*(M))) \geq 2.$$

Q.E.D.

LEMMA 11. Case (a) does not occur.

PROOF. Suppose (a). We have a pencil of <u>reducible</u> hyperplane sections $H = L_1' + L_2'$ on $\Phi(S)$. Let $\varpi: \tilde{S} \to S$ be a resolution of the base points of $|3K_S|$, and put $\tilde{\Phi} = \Phi \circ \varpi$. $\tilde{\Phi}^* H$ can be decomposed into $\tilde{L}_1 + \tilde{L}_2 + N$, where $\tilde{\Phi}(\tilde{L}_i) = L_i'$. $L_i = \varpi(\tilde{L}_i)$ is a

<u>movable</u> curve on S. So $L_iK_S \geq 2$. This leads to a contradiction:

$$3 = 3K_S^2 \geq (L_1+L_2)K_S \geq 4. \qquad \text{Q.E.D.}$$

LEMMA 12. Case (b) does not occur.

PROOF. Suppose (b). Let $\tilde{X} \xrightarrow{\varpi} X$ be the resolution of the base points and let $|L|+E_1+E_2+E_3=|3\varpi^*K_X|$, where $|L|$ is the moving part of $|3\varpi^*K_X|$ and the E_i are three distinct exceptional curves. $\tilde{\Phi} = \Phi_L$ is a finite morphism (see Lemma 9). Hence, in virtue of Zariski's Main Theorem, there exists the (holomorphic) involution ι of $\tilde{X}$ induced by $\tilde{\Phi}$. For the generic member $\hat{M}$ of $|2K_X|$, $\iota(\varpi^*(\hat{M})) \sim 2K_X$. On the other hand $\hat{\Phi}(\hat{M}) = 2H$, where H is a hyperplane section of $\Phi(X)$. Hence we have

$$6\varpi^*K_X - 2E_1 - 2E_2 - 2E_3 = \tilde{\Phi}^*(\hat{\Phi}(\hat{M})) = \varpi^*(\hat{M}) + \iota\varpi^*(M)) = 4\varpi^*K_X.$$

This is absurd. Q.E.D.

LEMMA 12. Case (c) does not occur.

PROOF. Suppose (c). If $\Phi(S)$ is not normal, $\Phi(S)$ contains a double line, and so we have a pencil of hyperplane sections $H=L'+L'_o$ where L'_o is the double line. We put $|3K_S| = \Phi^*H = L+L_1+L_2+G$, where $\Phi(L)=L'$ and $\Phi(L_i)=L'_o$. Since L is movable we have $LK_S \geq 2$ and $L_iK_S \geq 1$. This is impossible.

Next we assume that $\Phi(S)$ is a normal cubic. Let M_1 and M_2 be two generic members of $|2K_S|$. $\Phi(M_i)$ are sections of $\Phi(S)$ by quadric hypersurfaces. Let Λ denote the sublinear system of $|2H|$ generated by $\Phi(M_1)$ and $\Phi(M_2)$. For any $\lambda \in \Lambda$, $\Phi^*\lambda$ contains a divisor $M \in |2K_S|$ which is gererated by M_1 and M_2. This implies that for any $M \in |2K_S|$, $\Phi(M) \in \Lambda$. Since Φ is a finite morphism, the set of base points of Λ lies on the image of the base points of $|2K_X|$. Take a general point P_o of $\Phi(M)$. $\Phi^{-1}(P_o)=P+P'+P''$, P being a point of M. Let M' and M'' be two members of $|2K_X|$ which contain P' and P'' respectively. We have $\Phi(M)=\Phi(M')=\Phi(M'')$. In fact, if, say, $\Phi(M)\neq\Phi(M')$, then $P_o\in\Phi(M)\cdot\Phi(M')$ and P_o is a base point of Λ. From the discussions above, we see that there exists a natural holomorphic mapping $P^1\cong|2K_X|\cong|2K_S|\rightarrow P^1\cong\Lambda$ whose mapping degree $=3$. Since 3-sheeted covering $P^1\rightarrow P^1$ has its ramification locus of degree 4, the ramification locus $\mathcal{R}$ of Φ satisfies $\mathcal{R}\geq\cancel{4-2K_S=}4M$. On the other hand, since $\Phi(S)$ is a normal cubic surface, we have $K_S\geq\mathcal{R}-\Phi^*H$ $\geq 4M-3K_S$. This is a contradiction. Q.E.D.

LEMMA 13. Case (d) does not occur.

PROOF. Suppose (d). $|3K_X|$ has a unique base point P. Let $\tilde{X}\xrightarrow{\varpi}X$ be the quadric transformation at P, E the associated exceptional curve on X, and $\tilde{\Phi}$ the induced holomorhic mapping $\tilde{X}\rightarrow P^3$. Let $\hat{M}$ denote the generic member of $|2K_X|$. We have seen that $\hat{\Phi}(\hat{M})$ is a complete intersection of type (2,3). Therefore $|2H-\hat{\Phi}(\hat{M})|\neq\phi$ and <u>a fortiori</u> $|2(3K_X-E)-\varpi^*(\hat{M})|=$ $|2K_X-2E|\neq\phi$. Since P is a base point of $|3K_X|$, we have $H^1(X,\mathcal{O}_X(2E-2K_X))\neq 0$. This implies that $2K_S-2E$ is not 1-connected; i.e., there exist effective divisors D_1 and D_2 such that $D_1\sim D_2\sim K_S-E$, $D_1+D_2\in|2K_S-2E|$. $\tilde{\Phi}(D_i)$ is a <u>line</u> so $\dim|3K_S-E-D_1|\geq 2$. Let $N\in|3K_S-E-D_1|$. $N\sim 2K_S$. Choosing N generically, we may assume that $\tilde{\Phi}(N)$ is an irreducible plane curve of degree 3. Note that, since $\tilde{\Phi}|_{D_1}$ and $\tilde{\Phi}|_N$ is both double coverings, $\tilde{\Phi}^*(\tilde{\Phi}(D_1))=D_1$, $\tilde{\Phi}^*(\tilde{\Phi}(N))=N$. Moreover $\tilde{\Phi}(D_1)$ is a Cartier divisor on H, and $\mathcal{O}_{\tilde{\Phi}(N)}(\tilde{\Phi}(D_1))$ is an invertible sheaf whose degree = 3. ~~So we have~~

$$\mathcal{O}_N(D_1) = \tilde{\Phi}^* \mathcal{O}_{\Phi(N)}(\tilde{\Phi}(D_1))$$

This implies that deg $\mathcal{O}_N(D_1)=6$. But we have

$$ND_1 = 2K_X(K_X - E) = 2.$$

This is a contradiction. Q.E.D.

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